UDC 517.9

OPTIMAL CONTROL BY CERTAIN OBSERVATION PROCESSES

PMM Vol. 36, №2, 1972, pp. 181-188
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(Received June 3, 1971)

We study the question of optimal observation laws which ensure, to a specified accuracy, the determination of the unobservable coordinates. The conditions obtained for the solvability of the problems posed are stated in terms of the coefficients of the equations. The paper relates closely to [1, 2]. Another approach to an optimal observation problem under random perturbations occurs in [3, 4].

1. Let us assume that the state of an object at an instant t is defined by the phase coordinate vector $x(t) \in R_n$, where R_n denotes an n-dimensional Euclidean space, and the function $x(t)$, the solution of the system of ordinary differential equations

$$x(t) = Ax(t) + f(t), \quad x(0) = x_0 \quad (0 \leqslant t \leqslant T) \tag{1.1}$$

The vector $y(t)$ available for observation is such that

$$dy(t) = h(t) Hx(t)\, dt + \sigma d\xi(t), \quad y(0) = 0 \tag{1.2}$$

We invariably assume the fulfillment of the following constraints regarding the coefficients of Eqs. (1.1), (1.2). The matrix A of dimension $n \times n$ with constant elements and the deterministic measurable bounded function $f(t) \in R_n$ are specified. The random variable $x(0)$ has a nondegenerate Gaussian distribution with the parameters

$$m_0 = Mx(0), \qquad D_0 = M(x_0 - m_0)(x_0 - m_0)$$

Here the prime is the sign of transposition, M is the mean, and the matrix D_0 is positive definite in view of the nondegeneracy of the distribution of $x(0)$. The random Wiener process $\xi(t)$ is assumed independent of $x(0)$, the matrix σ is nonsingular, and the deterministic function $h(t)$ is scalar. The dimension of the observed quantity $y(t)$ may be arbitrary (from one to n). However, for convenience in what follows we take $y(t) \in R_n$, the dimension of the constant matrices H and σ is $n \times n$ and, moreover, the matrix σ is nonsingular.

The last requirement does not restrict generality. Indeed, if an i-dimensional quantity $y_i(t)$ is available for observation, i.e., Eqs. (1.2) have the form

$$dy_i(t) = h(t) H_i x(t)\, dt + \sigma_i d\zeta_i(t), \qquad y_i(0) = 0$$

then we should consider the new process $\zeta(t) = (\zeta_i'(t), \eta(t))$, where the Wiener process $\eta(t) \in R_{n-i}$ with independent components is independent of $\zeta_i(t)$ and of $x(0)$, and we should introduce new matrices H and σ in the following way. The first i rows of matrix H coincide with the corresponding i rows of matrix H_i, while the rest of the rows equal zero; the matrix σ_i occurs in the upper left corner of the matrix σ further, the elements $\sigma_{jj} = 1$ for $j = i + 1, \ldots, n$, and the other elements of matrix σ equal zero.

Control by the observation process is effected by means of choosing the scalar function $h(t)$. By q we denote a nonzero vector from R_n and we introduce into consideration the linear combination $q'x(T)$. As a result of carrying out the observations the error in the determination of $q'x(T)$ must be no larger than a specified fixed number $\alpha > 0$. To be precise, the inequality

$$q'M(x(T)-m(T))(x(T)-m(T))'q = q'D(T)q \leqslant \alpha \tag{1.3}$$

must hold, where $m(T)$ is the conditional mean of vector $x(T)$ under the conditions $y(s)$ $(0 \leqslant s \leqslant T)$, while the variance matrix $D(t)$ of the a posteriori distribution is given by the relations [1]

$$D^{\cdot}(t) = AD(t) + D(t)A' - D(t)H'(\sigma\sigma')^{-1}HD(t)\gamma(t)$$
$$\gamma(t) = h^2(t), \qquad D(0) = D_0 \qquad (0 \leqslant t \leqslant T) \tag{1.4}$$

Sometimes, in order to stress the dependence of the solution of problem (1.4) on the parameters defining it, it will be noted by the symbol $D(t, D_0, \gamma)$. Depending on the requirements imposed on the function $h(t)$, various formulations of the optimization problem for the observation process are possible [1], some of which are studied below.

2. Problem 1. Find a nonnegative square-integrable function $\gamma(t)$ which minimizes the integral

$$J(\gamma) = \int_0^T \gamma^2(t)\,dt \tag{2.1}$$

and is such that the function $D(T, D_0, \gamma)$ satisfies the estimate (1.3). Here the fixed constant $T < \infty$.

Theorem 2.1. For the solvability of Problem 1 with any number $\alpha > 0$, vector $q \in R_n$ and positive-definite matrix D_0, it is necessary and sufficient that the rank of the matrix

$$[H'(\sigma')^{-1}, A'H'(\sigma')^{-1}, \ldots, (A')^{n-1}H'(\sigma')^{-1}] \tag{2.2}$$

be complete (i.e., equal to the number n, the dimension of system (1.1)). In formula (2.2) the term A^n denotes the nth power of matrix A.

We preface the proof of Theorem 2.1 with an auxiliary lemma. We say that a function $\gamma(t) \geqslant 0$ is admissible if $J(\gamma) < \infty$ and the matrix $D(T, D_0, \gamma)$ satisfies relation (1.3).

Lemma 2.1. Let an admissible function $\gamma_1(t)$ exist. Then there also exists an optimal function solving Problem 1.

The proof of Lemma 2.1 is essentially very similar to the proof of the analogous assertion in the linear time-optimal problem ([5], Chap. 3). The assertion of Lemma 2.1 is obvious for the case when the number of admissible functions is finite. We now introduce into consideration the sequence $\gamma_i(t)$ $(i = 1, 2, \ldots)$ of admissible functions such that

$$\lim_{i\to\infty} J(\gamma_i) = J_0 = \inf J(\gamma) \tag{2.3}$$

where the "infimum" in the right hand side of (2.3) is computed over the set of all admissible functions. It is clear that the sequence $\gamma_i(t)$ belongs to some sphere in the Hilbert space of scalar functions on the interval $[0, T]$ and therefore is weakly compact ([6], p. 212). For simplicity of writing we take it that the sequence $\gamma_i(t)$ itself converges weakly to a function $\gamma_0(t)$. Hence, on the basis of [6] (p. 217) and of equality (2.3), we conclude that $J(\gamma_0) \leqslant J_0$. Furthermore, by a verbatim repetition of the arguments in [5] (pp. 143-145), we obtain that $\gamma_0(t) \geqslant 0$ $(0 \leqslant t \leqslant T)$. Hence, to prove the optimality of the function $\gamma_0(t)$ it suffices to show that the matrix $D(T, D_0, \gamma_0)$ satisfies inequality (1.3). However, this follows immediately from the estimates

$$q'D(T, D_0, \gamma_i)\, q \leqslant \alpha \qquad i = 1, 2, \ldots$$

from the weak convergence of sequence $\gamma_i(t)$, and from the formula [1]

$$D(T) = z(T)\left[D_0^{-1} + \int_0^T \gamma(t)\, z'(t)\, H'(\sigma\sigma')^{-1} Hz(t)\, dt\right]^{-1} z'(T) \tag{2.4}$$

in which D_0^{-1} is the inverse of matrix D_0, and $z(t)$ is the fundamental solution of the system of Eqs. (1.1), equal to

$$z(t) = \exp\left(\int_0^t A\, ds\right)$$

Lemma 2.1 is proved.

Proof of Theorem 2.1. Sufficiency. Let us show that an admissible function $\gamma(t)$ exists when the hypotheses of Theorem 2.1 are fulfilled. First of all we note that if

$$q'z(T)\, D_0 z'(T)\, q \leqslant \alpha$$

then Theorem 2.1 is already proved by virtue of formula (2.4), since in this case we can take $\gamma(t) \equiv 0$ as the admissible function. Therefore, in what follows we assume that

$$q'z(T)\, D_0\, z'(T)\, q > \alpha \tag{2.5}$$

Further, in view of Eq. (1.4) and of the positive definiteness of matrix D_0 it is not difficult to establish the positive definiteness of the matrix $D(t)$ $(0 \leqslant t \leqslant T)$ by using relation (2.4).

Suppose that the function $\gamma(t)$ in Eq. (1.4) is equal to a nonnegative constant ε for all $0 \leqslant t \leqslant T$. We set

$$\varphi(\varepsilon) = q'D(T, D_0, \varepsilon)\, q \tag{2.6}$$

On the basis of (2.4) - (2.6) we have

$$\varphi(0) > \alpha \tag{2.7}$$

Therefore, to prove the existence of an admissible function it is enough to establish the continuity of $\varphi(\varepsilon)$ and the relation

$$\lim_{\varepsilon \to \infty} \varphi(\varepsilon) = 0 \tag{2.8}$$

The rank of matrix (2.2) equals n and therefore the vector-valued functions which are the columns of the matrix $z'(t)\, H'(\sigma')^{-1}$ are linearly independent on the interval $0 \leqslant t \leqslant T$ (see [3], Sect. 19). Hence, the matrix

$$z'(T)^{-1} \int_0^T z'(t)\, H'(\sigma\sigma')^{-1} Hz(t)\, dt\, z(T)^{-1} \tag{2.9}$$

is positive definite. Consequently, there exists a nonsingular real matrix Q which simultaneously reduces matrix (2.9) to the unit matrix and the matrix

$$z'(T)^{-1}\, D_0^{-1}\, z(T)^{-1} \tag{2.10}$$

to diagonal form. Therefore, denoting by λ_i the eigenvalues of matrix (2.10), positive in view of the positive definiteness of matrix (2.10), we obtain on the basis of (2.4), (2.6) that the function $\varphi(\varepsilon)$ can be represented in the form

$$\varphi(\varepsilon) = \sum_{i=1}^{n} \beta_i (\lambda_i + \varepsilon)^{-1}$$

Here the β_i denote the diagonal elements of the matrix $Qqq'Q^{-1}$. Hence follow the continuity of the function $\varphi(\varepsilon)$ and the validity of the limit relation (2.8). By the same token the sufficiency of the requirements of Theorem 2.1 has been established.

Necessity. Let us assume the contrary, i.e., that the rank of matrix (2.2) is equal to $m < n$. Then we can find a nonzero vector q_1 which is orthogonal to all the columns of matrix (2.2). Therefore (see [3], Sect. 19), the function

$$\sigma^{-1} Hz(t)\, q_1 = \sigma^{-1} H \exp(At)\, q_1 \equiv 0 \qquad 0 \leqslant t \leqslant T \tag{2.11}$$

We now fix a certain number $\alpha > 0$ and a positive definite matrix D_0 and we define a vector q by the equality

$$q = z'(T)^{-1} D_0^{-1} q_1 \varepsilon$$

where ε is some constant. Let us select this constant ε in such a way that

$$q' z(T) D_0 z(T)\, q = q_1' D_0^{-1} q_1 \varepsilon^2 > \alpha \tag{2.12}$$

The latter is possible since $q_1' q_1 > 0$ by definition.

By the hypotheses of Theorem 2.1 Problem 1 is solvable for the number α the vector q and the matrix D_0. In other words, there exists a $\gamma(t)$ such that the estimate

$$\alpha \geqslant q' D(T, D_0, \gamma)\, q$$

is valid. But for any positive definite matrix $D(T)$ and any vector $q \in R_n$ there holds the equality

$$q' D(T)\, q = \max_{y \in R_n} [2y'q - y' D(T)^{-1} y] \tag{2.13}$$

Thus

$$\alpha \geqslant \max_{y \in R_n} [2y'q - y' D(T, D_0, \gamma)^{-1} y] \tag{2.14}$$

Note that the maximum over $y \in R_n$ of the expression

$$2y'q - y'z'(T)^{-1} D_0^{-1} z^{-1}(T)\, y$$

exists by virtue of the positive definiteness of the matrix $z'(T)^{-1} D_0^{-1} z^{-1}(T)$, is reached for

$$y = z(T) D_0 z'(T)\, q \tag{2.15}$$

and equals the left-hand side of expression (2.12). However, on the basis of (2.11) and of the definition of vector q

$$\int_0^T y'z'(T)^{-1} \gamma(t)\, z'(t) H' (\sigma\sigma')^{-1} H z(t)\, dt\, z^{-1}(T)\, y = 0$$

for a value of y equal to the right-hand side of Eq. (2.15). Thus, by virtue of formulas (2.14), (2.4), (2.12)

$$\max_{y \in R_n} [2y'q - y D(T, D_0, \gamma)^{-1} y] = q' z(T) D_0 z'(T)\, q > \alpha$$

which is impossible because it contradicts inequality (2.14). Theorem 2.1 is completely proved.

Remark. For any nonsingular matrix σ the rank of matrix (2.2) equals the rank of the matrix $(H', A'H', \ldots, (A')^{n-1} H')$. Thus, the requirements of Theorem 2.1 coincide with the condition for complete observability in a deterministic optimal observation problem (see [3], Sect. 30 - 38).

3. Example 1. Let the scalar equations (1.1), (1.2) have the form

$$x^{\cdot}(t) = ax(t) + f(t)$$
$$dy(t) = h(t)\, H\, x(t)\, dt + d\xi(t), \quad H = \text{const} \neq 0 \tag{3.1}$$

Then the variance $D(t)$ is defined by the relations

$$D^{\cdot}(t) = 2a\, D(t) - D^2(t)\, H^2 \gamma(t), \quad D(0) = D_0 \tag{3.2}$$

and inequality (1.3) becomes the requirement $D(T) \leqslant \alpha q^{-2}$. Hence from (3.2) and (2.4) it follows that the optimal function $\gamma(t) \equiv 0$ for $q^2 D_0 e^{2aT} \leqslant \alpha$. Now let $q^2 D_0 e^{2aT} > \alpha$ Using the method of undetermined Lagrange multipliers and the explicit form of solution (2.4) of Eq. (3.2) we obtain that the optimal value $\gamma_0(t)$ is

$$\gamma_0(t) = 4a(\alpha^{-1}q^2 - D_0^{-1}e^{-2aT})(e^{4aT} - 1)^{-1} e^{2a(t+T)} H^{-2}$$

It is also obvious that the coefficients of Eqs. (3.1) satisfy the requirements of Theorem 2.1.

Corollary 1. Let there be given m numbers t_i such that $0 < t_1 < t_2 < \ldots$, $\ldots < t_m = T$ and m vectors $q_i \in R_n$. We are required to choose a function $\gamma(t) \geqslant 0$ minimizing integral (2.1) and satisfying the estimates

$$q_i' D(t_i, D_0, \gamma)\, q_i \leqslant \alpha_i \qquad (i = 1, \ldots, m)$$

where α_i are specified positive constants. By repeating with insignificant changes the proof of Theorem 2.1 we obtain that the necessary and sufficient condition for the solvability of the problem posed coincides with the necessary and sufficient condition for the solvability of Problem 1, established in Theorem 2.1.

Corollary 2. Let us derive the sufficient conditions for the solvability of Problem 1 for certain systems of form (1.1), (1.2) with variable coefficients. Precisely speaking, we assume that Eqs. (1.1), (1.2) have the form

$$x^{\cdot}(t) = A(t)\, x(t) + f(t), \quad x(0) = x_0 \quad (0 \leqslant t \leqslant T)$$
$$dy(t) = h(t)\, H(t)\, x(t)\, dt + \sigma(t)\, d\xi(t), \quad y(0) = 0 \tag{3.3}$$

where the matrices $A(t)$, $H(t)$, $\sigma(t)$ with measurable bounded elements for all $0 \leqslant t \leqslant T$ satisfy the requirements in Sect. 1. If, in addition, we can indicate a point s on the interval $[0, T]$ such that:

1) the derivatives of the matrices $A(t)$ and $\sigma^{-1}(t)\, H(t)$ upto order $n - 1$ exist and are continuous in some neighborhood of point s;

2) at point s the rank of the matrix

$$(K_1(s), \ldots, K_n(s))$$

where

$$K_1(s) = H'(s), (\sigma'(s))^{-1}, \qquad K_{i+1}(s) = \frac{dK_i(s)}{ds} + A'(s)\, K_i(s)$$

equals the number n, then Problem 1 for system (3.3) is solvable for any number $\alpha > 0$, vector $q \in R_n$ and nonsingular matrix D_0

The proof of Corollary 2 is completely analogous to the proof of Theorem 2.1 with the sole difference that this time the positive definiteness of the matrix

$$\int_0^T z'(t)\,H'(t)\,(z(t)\,z'(t))^{-1}\,H(t)\,z(t)\,dt$$

corresponding to matrix (2.9) in the proof of Theorem 2.1, is established with the aid of the results in [3] (Sect. 20).

4. In this section we consider an optimal observation problem in which it is also required to ensure the fulfillment of inequality (1.3) under assumptions on the function $h(t)$, different from Problem 1. The equations of motion and of observation have form (3.1).

Problem 2. Let the observation time be finite but not fixed. The observations cease at the first instant T at which inequality (1.3) is fulfilled. It is required to determine a piecewise-constant function $h(t)$, taking two values (either zero or unity), minimizing the integral

$$\int_0^T h(t)\,dt \tag{4.1}$$

such that

$$q' D(T, D_0, h)\, q = q' D(T, D_0, \gamma)\, q \leqslant \alpha \tag{4.2}$$

Here the equality $h(t) = 0$ signifies that observations are not made at the instant t (by virtue of the independence of the distribution of $x(0)$ from the distributions of process $\xi(t)$). Everywhere in this section $h(t)$ denotes a function satisfying the constraints just formulated, while $N(q, \alpha)$ denotes the set of all positive-definite matrices D such that $q' Dq > \alpha$.

Theorem 4.1. Let there exist a nonnegative continuous function $\omega(D)$ of the matrix argument D, for which the total time derivative $\dot{\omega}(D)$, taken relative to Eqs. (3.2) with $h(t) \equiv 1$, is piecewise continuous and, for some number $\varepsilon < 0$ satisfies the estimate

$$\dot{\omega}(D) \leqslant \varepsilon, \qquad D \in N(q, \alpha)$$

Then Problem 2 is solvable under any initial condition D_0.

Proof. Similarly to Sect. 2 we call the function $h(t)$ an admissible observation if integral (4.1) is finite and inequality (4.2) is valid for this $h(t)$. Further, analogously to the proof of Lemma 2.1, it is not difficult to establish that from the existence of an admissible observation, the existence of a function $u_0(t)$ ensues which solves the following problem. Find a function $u(t)$ such that $0 \leqslant u(t) \leqslant 1$, minimizing integral (4.1), where inequality (4.2) is fulfilled for $\gamma(t) = u(t)$. From this and from the maximum principle it follows that for any t the function $u_0(t)$ equals either zero or unity, i.e., $u_0(t)$ is the optimal observation law also for Problem 2.

We note further that on the basis of (2.4), (2.13),

$$q' D(T, D_0, 1)\, q \leqslant q' D(T, D_0, h)\, q$$

for any T and for observation $h(t)$. Thus, Problem 2 has a solution if

$$q' D(T, D_0, 1)\, q \leqslant \alpha \tag{4.3}$$

for some T. It is clear that it is sufficient to establish estimate (4.3) only under the additional assumption $D_0 \in N(q, \alpha)$.

Let us now assume that for all $t > 0$ the variance

$$D(t, D_0, 1) \in N(q, \alpha)$$

Then for some finite s the function $\omega(D(s, D_0, 1)) < 0$, which is impossible since it contradicts the requirement $\omega(D) \gg 0$. Theorem 4.1 is proved.

Example 2. We consider the scalar Eqs. (3.1) of motion and observation. Let us find the conditions in terms of the coefficients of these equations under whose fulfillment Problem 2 is solvable. In this example the set $N(q, \alpha)$ is the halfline $D > \alpha q^{-2}$. Hence, from the positiveness of the variance $D(t)$ for any finite t and from (1.4) it follows that we can take $D(t)$ as the function $\omega(D)$. Consequently, on the basis of Theorem 4.1, Problem 2 for system (3.1) is solvable for $a < {}^1/_2 H^2\alpha q^{-2}$. The latter condition for the solvability of Problem 2 can also be obtained directly from an analysis of Eqs. 1.4.

Remark. Problem 2 for the system (1.1), (1.2) is solvable if all the eigenvalues of matrix A have negative real parts. This follows immediately from the formula

$$q'D\ (T,\ D_0,\ h)q \leqslant q'D\ (T,\ D_0,\ 0)q$$

and from Eqs. (1.4), (2.4). However, as Example 2 shows, the stability of matrix A is not a necessary condition for the solvability of Problem 2. Let us illustrate what we have said by a simple example. In Eq. (1.1) let matrix A be symmetric and matrix H nonsingular. By λ_0 we denote the largest eigenvalue of matrix A and by $\lambda_1 > 0$ the smallest eigenvalue of the matrix $V = H'\ (\sigma\sigma')^{-1}H$. Further, with the aid of a nonsigular real transformation we reduce matrix A to diagonal form and matrix V to normal form. Then, using further the expressions (2.4), (2.13) we obtain that Problem 2 for system (1.1), (1.2) is solvable for $2nq'q\lambda_0 < \alpha\lambda_1$. We note, finally, that if all the eigenvalues of matrix A are real and Problem 2 for system (1.1), (1.2) is solvable for $h_0(t)$, then, analogously to the proof of Fel'dbaum's theorem ([5], p.134), it is not difficult to establish with the aid of (1.4) that $h_0(t)$ has no more than n switching points.

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Translated by N. H. C.